COORDINATING PEBBLE MOTION ON GRAPHS, THE DIAMETER OF PERMUTATION GROUPS AND APPLICATIONS

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Coordinating Pebble Motion on Graphs, the Diameter of Permutation Groups, and Applications

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Abstract

The problem of memory management in totally distributed computing systems leads to the following movers' problem on graphs:

Let G be a graph with n vertices with $k < n$ pebbles numbered $1,\ldots,k$ on distinct vertices. A move consists of transferring a pebble to an adjacent unoccupied vertex. The problem is to decide whether one arrangement of the pebbles is reachable from another, and to find the shortest sequence of moves when it is possible.

In the case that G is biconnected and $k = n - 1$, Wilson (1974) gave an efficient decision procedure. However, naive implementation of his proof gives exponentially long move sequences for solutions. We generalize the decision procedure to all graphs and any number of pebbles. Further, we prove matching $O(n^3)$ upper and lower bounds on the number of moves required, and show how to efficiently plan solutions.

It is hoped that the algebraic methods introduced for the graph puzzle will be applicable to special cases of the general geometric movers' problem, which is PSPACE-hard (Reif (1979)).

We consider the related question of permutation group diameter. Driscoll and Furst (1983) obtained a polynomial upper bound on the diameter of permutation groups generated by cycles of bounded length. Their results do not apply to arbitrary length cycles. We obtain the following partial extension of their result to unbounded cycles:

If G (on n letters) is generated by cycles, one of which has prime length $p < 2n/3$, and G is primitive, then $G = A_n$ or S_n and has diameter less than $2^{6\sqrt{p}+4}n^8$. This is a moderately exponential bound.

1. Introduction

The management of memory in totally distributed computing systems is an important issue in hardware and software design. On an existing hardware network of devices, there is the problem of how to coordinate the transfer of one or more indivisible packets of data from device to device without ever exceeding the memory capacity of a device. Depending on the severity of the memory capacity, a considerable number of intermediate transfers may be necessary to clear a "path" for the movement of a data packet along a network. A combination of almost filled devices and a network configuration with few paths can, in fact, make impossible the transfer of the data packets intact.

Suppose we consider a simplified version of the above problem, where each device has unit capacity and each packet occupies one unit of memory. Then at any moment in time, any given device is either empty or is totally filled. Suppose also that at any time each data packet resides in some device. It is also assumed that only one packet may be moved at a time, from its current device to any empty immediately adjacent device. Under these assumptions, it is interesting to know whether it is possible to start from one given distribution of the packets in the network, and end with another given rearrangement, and to know how many moves are required when the rearrangement is possible.

This version of the network problem immediately translates into the following movers' problem on graphs:

Let G be a graph with n vertices with $k < n$ pebbles numbered $1,...,k$ on distinct vertices. A move consists of transferring a pebble to an adjacent unoccupied vertex. The problem is to decide whether one arrangement of the pebbles is reachable from another, and to find the shortest sequence of moves to find the rearrangement when it is possible.

It is seen that this latter problem is a generalization of Sam Loyd's famous "15-puzzle". In this puzzle, 15 numbered unit squares are free to move in a 4x4 area with one unit square blank. The problem is to move from one arrangement of the squares to another. One can easily show that this puzzle is equivalent to the graph puzzle on the square grid in figure 1, with 15 numbered pebbles on the vertices and one blank vertex.

In the case that G is biconnected and $k = n - 1$, Wilson (1974) gave an efficient decision procedure. However, a naive implemenation of his proof yields solutions requiring exponentially many moves. His approach involved deriving a 3-cycle and 2-transitivity. This basis is known to generate all possible even permutations on the pebbles, but it is not clear whether the basis is efficient. We avoid this difficulty by deriving 3-transitivity. This is trickier to prove than 2-transitivity, but it enables us to generate all even permutations efficiently . In this way, an $O(n^3)$ upper bound is obtained for the number of moves required in the Wilson case.

Then we generalize the decision procedure to all graphs and any number of pebbles, and we show that again at most $O(n^3)$ moves are needed and can be efficiently planned.

Finally, we find an infinite family of graph puzzles for which it is proved that $O(n^3)$ moves are needed for solutions. Thus the upper and lower bounds match to within a constant factor.

A topic of related interest is the subject of permutation groups and their diameter with respect to a set of generators. Briefly, the diameter of a permutation group G with respect to a set S of generators for G is defined to be the smallest positive integer k such that all elements of G are expressible as products of the generators of length at most k.

Consideration of the pebble coordination problem leads naturally to questions about permutation groups. Consider the graph in figure 2, with vertex v blank and pebbles $a_1, \ldots, a_t, c_1, \ldots, c_r, b_1, \ldots, b_s$, and y on the other vertices. It is seen that any sequence of moves from this position will, upon the first return of the blank to v, net one of the following permutations on the pebbles: $A = (c_1 c_2 \ldots c_r y a_t \ldots a_2 a_1)$ or $B = (y c_r \ldots c_2 c_1 b_1 b_2 \ldots b_s)$ or $C = (b_1 b_2 \ldots b_s y a_t \ldots a_2 a_1)$ or A^{-1}, B^{-1}, C^{-1} or the identity permutation. Hence the set of rearrangements of the pebbles (with v blank) is the group of permutations generated by $S = \{A, B, C, A^{-1}, B^{-1}, C^{-1}\}$. Deciding whether a rearrangement is solvable amounts to testing membership of the corresponding permutation in the group generated by S; minimum number of moves is clearly related to the shortest product of generators yielding the permutation.

We view the introduction of algebraic methods as useful for the solution of movers' problems. Whereas general geometric movers' problems are PSPACE-hard (Reif (1979)), it is hoped that the techniques introduced for the solution of the pebble coordination problem may be applicable to special cases of the general geometric problem.

We now briefly discuss the state of the art in permutation group membership and diameter questions. Furst, Hopcroft and Luks [FHL] give a polynomial time algorithm for deciding whether a given permutation g is in $G(S)$, the group generated by S. Thus the analogue of the graph decision problem is in P. One also immediately has a P-time criterion for deciding solvability of the Rubik's Cube and the Hungarian Rings puzzles. The situation is not as fortunate when one tries to find the length of the shortest generator sequence for a given permutation: Jerrum [J] has recently shown this to be PSPACE-complete! The difficulty may be related to the fact that some groups may have superpolynomial diameter. For example, the group G generated by the single permutation (12)(345)(6789 10)...(...s) where s is the sum of the first n prime numbers, can be shown to have diameter roughly on the order of $2^{O(\sqrt{n})}$. This contrasts with the analogous question for the pebble coordination problem, where no solution can ever require more than $O(n^3)$ moves. Therefore the group diameter question is in some sense more general, and probably more difficult, than the corresponding question for pebble motion.

There are nonetheless some interesting recent results concerning upper bounds on group diameter, for special generating sets. Driscoll and Furst [DF] have shown that if all the generators are cycles of bounded length, then the group has $O(n^2)$ diameter where n is the number of letters that the group acts on. More recently, McKenzie [M]

obtained the upper bound $O(n^k)$ on diameter for groups, each of whose generators moves at most k letters. This is polynomial if k is bounded.

The foregoing results leave open the question of a group's diameter when the generators are arbitrary (not of bounded length) cycles. In chapter 3 we informally discuss certain generalizations of the Hungarian Rings puzzle, and find sufficient conditions for the required number of moves to be polynomial. Examples which do not meet these sufficient conditions are offered as possible candidates for groups with superpolynomial diameter. The rest of chapter 3 consists of a number of new results in permutation groups, which extend classical theorems by providing upper bounds on diameter. We obtain the following theorem as a corollary:

If G (on n letters) is generated by cycles, one of which has prime length $p < 2n/3$, and G is primitive, then $G = A_n$ or S_n and has diameter less than $2^{6\sqrt{p}+4}n^8$.

This is a moderately exponential upper bound, but is nonetheless superpolynomial. It remains of interest to know whether the bound can be significantly improved, or whether the diameter really can be this large.

At the end of the paper we present conjectures, open problems, and suggestions for further research in movers' problems and permutation group diameter.

2. Coordinating Pebble Motion on Graphs

In this chapter we will solve the pebble coordination problem given in the introduction:

Let G be a graph with n vertices with $k < n$ pebbles numbered 1,...,k on distinct vertices. A move consists of transferring a pebble to an adjacent unoccupied vertex. The problem is to decide whether one arrangement of the pebbles is reachable from another, and to find the shortest sequence of moves to find the rearrangement when it is possible.

We make the assumption that the set of occupied vertices of G is the same in both the initial and final positions. Then two positions define a permutation on the pebbles in a natural way, and so we can readily introduce the methods of group theory. There is no loss of generality, as we can show how to efficiently convert a puzzle into this form.

We also assume that all graphs are *simple*, that is, no two vertices are directly joined by more than one edge, and no vertex is joined to itself by an edge. It is clear that if a graph G is nonsimple, we can remove the "extraneous" edges to get a simple graph G', and the graph puzzle on G' is exactly equivalent to that on G, both with respect to solvability and the number of moves needed to solve it. Hence there is no loss of generality in making this assumption.

It turns out to be natural to divide the analysis into two cases: 1. Biconnected graphs with all but one vertex occupied (the Wilson case) 2. All other cases (unconnected graphs, separable graphs, and the biconnected case with at least two blanks).

The Wilson case is the more interesting from the group theoretical point of view. It can be shown that graphs without closed paths need at most $O(n^2)$ moves to solve; it is the loop structure of biconnected graphs which results in some puzzles requiring $O(n^3)$ moves.

2.1. Biconnected graphs, one blank

We introduce Wilson's theorem, and prove it in a way which we believe is simpler than the original proof. This new proof enables us to obtain a $O(n^3)$ upper bound on the number of moves needed for solution.

Let G_0 be the graph consisting of two vertices joined by an edge. Define a *polygon* to be a graph consisting of a simple closed path containing at least two vertices (where a simple closed path is a path from a vertex to itself which visits no intermediate vertex more than once). A polygon looks like a "loop" containing two or more vertices. Let T_0 be the graph shown in figure 3.

Theorem 1 (Wilson)

Let G be a biconnected graph on n vertices, other than a polygon or T_0, with one blank vertex. If G is not bipartite, then the puzzle is solvable. If G is bipartite, then the puzzle is solvable iff the permutation induced by the initial and final positions is even.

Since bipartitism can be tested in polynomial time, Wilson's criterion is polynomial time.

For G a polygon, only cyclical rearrangements of the pebbles are possible, so it is easy to check reachability in this case. For the special graph T_0, we can simply precalculate (by exhaustive search) a table of all pairs of positions, indicating which pairs are mutually reachable. Table lookup is constant time, hence we have a P-time decision algorithm for all biconnected graphs with one blank.

It will turn out as a special case of the next section, that the biconnected case with two or more blanks is always solvable.

Sketch of Proof

(A complete proof will be given in the final version.)

It is a well-known fact in graph theory that a biconnected graph, other than G_0, can be viewed as being "grown", by starting with a polygon graph and successively adding zero or more "handles" (a handle is a simple path with 0 or more internal vertices). A biconnected graph which can be "grown" by adding i handles to a polygon, appears pictorially to consist of $i + 1$ simple loops joined together in some way. This number of loops is called the *Betti number* of the graph. We will often denote a biconnected graph with Betti number i by the term "T_i-graph". Wilson's theorem will be proved by induction on the Betti number of the graph. We skip the T_1-graphs (the polygons) and begin the induction with the T_2-graphs (except T_0).

The main step is to show that the group of possible induced permutations always contains the alternating group A_{n-1} on the $n-1$ pebbles. The final step is to determine whether the group is A_{n-1} or S_{n-1}. The group will be S_{n-1} iff it contains an odd

permutation, and it is easy to see that there is an odd permutation iff the graph has a closed path of odd length. As a graph has a closed path of odd length iff it is not bipartite, we see that the group is A_{n-1} if the graph is bipartite, and S_{n-1} if the graph is not bipartite. Therefore, to check solvability on a bipartite graph, it is necessary and sufficient that the induced permutation be even; on a nonbipartite graph, the puzzle is always solvable.

To show that the group of induced permutations contains the alternating group, we show how to obtain a 3-cycle and how to obtain 3-transitivity. From this, the alternating group is efficiently generated.

A 3-cycle is obtained roughly as follows. A T_2-graph looks like that pictured in figure 2. Assume first that $r = 0$, i.e. the center arc has no internal vertices. $A = (ya_t...a_1)$ and $B = (b_1...b_sy)$ are permutations induced by moving pebbles around, respectively, the left or right loops. Then $ABA^{-1}B^{-1} = (yb_sa_1)$, a 3-cycle. If $r > 0$, then $ABA^{-1}B^{-1}$ is a product of two swaps; we will show in the full proof how to obtain a 3-cycle from this, if the graph induces 4-transitivity. It turns out that the graph T_0 is the only T_2-graph which does not induce a 3-cycle. We then show that all T_i-graphs, $i > 2$ give a 3-cycle, because they are formed by adding handles to a T_2-graph which can induce the 3-cycle. The hole in the induction due to T_0 will be taken care of with no difficulty.

3-transitivity will also be shown by induction. It will be shown that all T_2-graphs except the graph G_1 shown in figure 4 are 3-transitive, by a simple lemma. Then we show how adding a handle to a 3-transitive graph yields a 3-transitive graph. The hole in the induction due to G_1 will be handled without trouble.

Putting 3-cycle and 3-transitivity together, we will conclude that all T_i-graphs, $i >= 2$, generate at least the alternating group, except T_0.

In Wilson's proof, a 3-cycle and 2-transitivity are derived. This basis is known to generate A_n, but it is not clear whether it does so efficiently. For this reason, we have established 3-transitivity. Conjugation easily gives all 3-cycles, and so A_n efficiently. Our proof of 3-transitivity is slightly tricky, especially in dealing with graph G_1, but it guarantees that A_n is generated efficiently.

Theorem 2

Let G be a biconnected graph. Let $n = |V(G)|$. If labeling g can be reached from labeling f at all, then this can be done within $O(n^3)$ moves, and such a sequence of moves can be efficiently generated.

Sketch of proof (details in final version)

We can show that a 3-cycle can always be obtained in $O(n^2)$ moves (either $ABA^{-1}B^{-1}$ gives a 3-cycle in $O(n)$; or we get a product of two swaps, in which case we can do 4-transitivity in $O(n^2)$ moves to get a 3-cycle), and that 3-transitivity requires at most $O(n^2)$ moves. Then by conjugation we obtain any 3-cycle within $O(n^2)$ moves. Since any element of A_n is a product of $O(n)$ 3-cycles, the total for A_n is $O(n^3)$. If the group is S_n, then any permutation is a product of an odd permutation and an element of A_n. An odd permutation is generated by a closed path of odd length in $O(n)$ moves. Hence S_n also requires at most $O(n^3)$ moves.

2.2. Separable graphs, and nonseparable graphs with > 1 space

The basic element which distinguishes separable graphs from biconnected graphs is the existence of isthmuses (of length $>= 0$), which if severed will separate the graph. One can think of a separable graph as being a tree, or a tree structure connecting one or more biconnected graphs (see figure 5 for an example).

Much of what follows is motivated by the example shown in figure 6. This graph consists of a simple nonclosed path of length m which connects subgraphs A and B. Suppose we wish to move pebble T from v to w. Since A has no blank vertices, it is clear that T can reach w if and only if B has $m-1$ or more blank vertices. Therefore, the number of blanks has a direct effect on the ability of pebbles to cross isthmuses. Conversely, the lengths of the isthmuses will determine whether or not certain pebbles can cross from one component into another.

It turns out that we can naturally divide a graph G in this way into subgraphs G_i connected by isthmuses, with the property that pebbles can move anywhere within each G_i but cannot leave G_i. Each pebble in its initial position is assigned in a natural way to the G_i (if any) to which it is confined, otherwise it is confined to an isthmus. This decomposition induces subpuzzles on the G_i's and their pebbles, and it will be shown that the original puzzle is solvable iff all the subpuzzles are solvable and the tokens trapped on isthmuses do not change order. Figure 7 shows the G_i subgraphs for the graph of figure 5 (exactly how the G_i are determined will be explained in the final version).

The final step in the analysis is to study solvability of subpuzzles on the G_i. When there is one blank, the G_i's turn out to be biconnected and so the Wilson criterion applies. We will show that when there are two or more blanks, the G_i subpuzzles are always solvable (subject to the condition that the G_i contains the same pebble set before and afterwards). Informally, one reason is that the G_i's were defined in such a way that pebbles can cross all isthmuses in G_i, and so get from any vertex to any other vertex (see figure 8 for an illustration); we will show how to achieve 2-transitivity in this way, by moving one pebble after another to its destination. The other reason is that two blanks are sufficient to achieve a swap of a pair of pebbles near a vertex of valence three (see figure 9). Combining 2-transitivity and the swap yields all permutations of the pebbles.

The General Criterion

Here is an outline of the general criterion. Details of how to determine the subpuzzles and the pebbles confined to isthmuses will be given in the final version.

Let G be a graph with k tokens and $m = n - k$ blanks. Let f, g be the starting and ending positions. Move blanks in f to form a labeling f' whose blanks are in the same locations as in g. Then f and g are mutually reachable iff f' and g are mutually reachable.

So without loss of generality assume that f and g have blanks in the same places.

If G is nonsimple, remove extra edges. This will neither hurt nor help solvability.

If G is not connected, check that the token partition induced naturally by the connected subgraphs is consistent, and that each connected subgraph puzzle is solvable.

If G is connected:

If G is nonseparable: if $m = 1$ use the criterion in Theorem 1; if $m > 1$, then the puzzle is solvable, unless G is a polygon, in which case only cyclic rearrangements are possible.

Otherwise, determine the subpuzzles and check that the pebble sets in each subpuzzle match before and after. Also determine the pebbles confined to isthmuses, and check that they are the same before and after, and in the same order. If all this is OK, then if $m = 1$ check that each subpuzzle is solvable (for $m = 1$, all components are nonseparable, so use Wilson's criterion). If $m > 1$, then each subpuzzle will be solvable, so we're OK in this case.

This completes the criterion.

It will be easy to show based on the analysis of case 1, that solutions with at most $O(n^3)$ moves exist and can be efficiently planned. In the final section of this chapter, we construct an infinite family of graph puzzles which are proved to require $O(n^3)$ moves for solution.

2.3. $O(n^3)$ lower bound

We now complement the above result with a lower bound which matches, to within a constant factor.

Theorem

There exists a constant $c > 0$ and an infinite sequence of graph puzzles Puz_i on increasingly large graphs G_i with n_i vertices, such that for each i, Puz_i requires at least cn_i^3 moves for solution.

Proof

Let Puz_i consist of graph G_i shown in figure 10, with $2i+1$ vertices and $2i$ pebbles, and starting and ending positions as shown. We will show that Puz_i requires $O(i^3)$ moves, as follows. A move sequence that does not waste moves (by retracing move sequences just made) is seen to consist of cycles A, B and their inverses, interspersed in some order (e.g. $ABAAAABA^{-1}B$). It would be wasteful to do B twice in succession, since this would cancel itself. Hence a move sequence can be represented by the form $A^{i_1}BA^{i_2}B...A^{i_k}BA^{i_{k+1}}$ where i_j is a nonzero integer (positive or negative), except i_1 and i_{k+1} may be 0.

Now consider the "entropy function" of position

$E = \sum_{j=0}^{i}$ (shortest circular distance from pebbles j to $j+i$)

where circular distance is either clockwise or counterclockwise. Initially, $E = i^2$; at the end, $E = i$. Change in E is $i^2 - i$.

It is seen that A does not change E, and B changes E by 0 or by 2. Hence to effect the change in E requires $O(i^2)$ occurrences of B in the move sequence. But because

occurrences of A^{i_2} and B alternate, this implies that A occurs at least $O(i^2)$ times. Since the number of moves to perform the cycle A is $O(i)$, we need at least $O(i^3)$ moves for solution.

This completes the proof of the lower bound.

3. The Diameter of Permutation Groups

As mentioned in the introduction, this chapter is concerned with the diameter of permutation groups generated by sets of cyclic permutations. We begin with some examples of generator sets which yield groups of polynomial diameter, then speculate on some conditions on the generator set which might give groups of superpolynomial diameter. The main part of the chapter consists of theorems which give information about the diameter of a group under various conditions. They imply the result given in the introduction, which is a moderately exponential upper bound on the diameter of groups generated by cycles which satisfy a few conditions.

3.1. What is not of exponential diameter, and what might be

The Hungarian Rings puzzle consists of two intersecting circular rings in which distinguished marbles circulate. The problem is to obtain a desired rearrangement of the marbles by a sequence of operations, where an operation consists of circulating the marbles in one of the rings. This problem immediately translates into the permutation problem of determining membership in the group generated by two intersecting cyclic permutations. By [HFL], we can decide membership in polynomial time; however, it is of interest to know how many "moves" are required, i.e. the length of the shortest word which gives the desired permutation.

In figure 11 is shown schematically two cyclic permutations which intersect at two points. This corresponds to the commercial version of the Hungarian Rings. Note that this is not like a pebble puzzle on a T_3-graph, because only A and B are possible, and not the third loop; the Hungarian rings is a physical movers' problem which imposes this restriction mechanically. This gives reason to expect that the number of moves may need to be larger in some permutation puzzles than in the pebble puzzles.

What is the diameter of the group generated by these two cycles? It is first useful to observe that, if some arc C contains at least r internal nodes, and an arc D on the other cycle contains at least one internal node, then we can get $r+1$-transitivity in $O(rn)$-long moves. This is done, roughly speaking, by moving one desired marble after another to a_1, then rotating it onto arc C. The cycle not containing arc C is rotated to bring the next desired marble to a_1, leaving the contents of C undisturbed. Arc D serves as temporary "storage" of a marble which, already on arc C, needs to be removed from C and then placed onto C at the right place.

Suppose that in the figure, $l >= 6$ and $m >= 1$. Then we have efficient 6-transitivity. Now $ABA^{-1}B^{-1} = P = (a_1 a_{k+l+m+q} a_{k+l})(a_k a_{k+1} a_{k+l+m})$. Using 6-transitivity, we can find a permutation P_1 which sends $a_1, a_{k+l+m+q}, a_{k+l}$ to

$a_1, a_{k+l}, a_{k+l+m+q}$ respectively and fixes a_k, a_{k+1}, a_{k+l+m}. Then conjugating P by P_1 gives $P_2 = (a_1 a_{k+l} a_{k+l+m+q})(a_k a_{k+1} a_{k+l+m})$. P_2 is a product of two 3-cycles, one the inverse of the one in P, the other the same as the other in P. So $PP_2 = (a_k a_{k+l+m} a_{k+1})$, a 3-cycle. Then, using 3-transitivity, we get the alternating group. Hence $l >= 6$ and $m >= 1$ implies a polynomial diameter for the Hungarian Rings puzzle with the rings intersecting at two places.

What happens if the number of intersection of the two cycles is some number k greater than 2? By similar reasoning to the above, we get $ABA^{-1}B^{-1}$ to be a product of k 3-cycles. Then a conjugation argument similar to the above yields that, if we have $3k$-transitivity, then we can get a single 3-cycle. How do we get efficient $3k$-transitivity? Well, an arc of $3k - 1$ nodes and another arc with one node would suffice. Or, in the case that k is bounded, then it is known [DF] that the existence of k-transitivity is enough to ensure k-transitivity in $O(n^k)$-long words, which is polynomial for fixed k. However if k is large, then this bound is exponential. If no arc has enough nodes in it, there might be no efficient way to get the desired degree of transitivity.

The foregoing considerations suggest that a good candidate for a Hungarian Rings puzzle with superpolynomial diameter is one with lots of crossings and no long arcs (see figure 12). To be more quantitative, suppose that there are k equally spaced crossings. Then the arcs have length on the order of n/k. We want this to be less than $3k$. So: $n/k < 3k$, i.e. $k > \sqrt{n/3}$. This suggests that we should use at least on the order of $\sqrt{n}$ crossings to create a likely exponential puzzle. It would be of great interest to establish an exponential or moderately exponential lower bound for some of these "candidate" puzzles.

We now leave these examples and speculations, and state some results about the diameter of permutation groups (proofs in final version).

3.2. Some results about the diameter of permutation groups

The following are classical theorems in the theory of permutation groups.

Theorem A

If the group G on n letters is k-transitive and $k > n/3 + 1$, then $G = A_n$ or S_n.

Theorem B

If G is primitive on n letters, and a subgroup H moves only $m < n$ letters and is primitive on them, then G is $n - m + 1$-transitive.

We prove the following versions of these theorems, which give information about the diameter:

Theorem 1

If group G on n letters is k-transitive in words of length $<= L$, the generator set S is closed under inverses, and $k > n/3 + 1$, then $G = A_n$ or S_n and $\mathrm{Diam}(G(S)) < 4n^2 L$.

Theorem 2

If G is primitive on n letters, and H is the primitive subgroup generated by a cyclic permutation of prime length $p < n$, and the generator set S is closed under inverses, then G is $n - p + 1$-transitive using words of length $< 2^{6\sqrt{p}+1}n^3(n^2 + diam(H(S)))$.

Theorem 3

If G is primitive on n letters, and H is a 2-transitive subgroup which moves only $2 <= m < n$ letters, and the generating set S is closed under inverses, then G is $n - m + 1$-transitive using words of length $< 2^{9\sqrt{p}+1}n^3(n^2 + diam(H(S)))$.

We were not able to prove an effective version of theorem B for arbitrary primitive H, but did obtain the special cases contained in theorems 2 and 3.

The following is an easy corollary.

Theorem

If a primitive group G on n letters is generated by a set S of cyclic permutations, one of prime length $p < 2n/3$, then G is A_n or S_n, and $\mathrm{Diam}(G(S)) < 2^{6\sqrt{p}+4}n^8$.

All proofs will be given in the final version.

This last theorem provides a partial extension of [DF]'s upper bound for bounded cycles to unbounded cycles. It would be desirable to generalize the result to apply to all cycles, and to find a matching lower bound on diameter.

4. Conclusion and Open Problems

We have obtained some results in pebble coordination problems and the diameter of permutation groups. Specifically, we derived:

1. An efficient decision algorithm for the general pebble coordination problem on graphs.

2. $O(n^3)$ matching upper and lower bounds on the number of moves to solve pebble coordination problems.

3. $2^{6\sqrt{p}+3}n^8$ upper bound on diameter of A_n or S_n when generated by cycles, one of which has prime length $p < 2n/3$.

We see 1. as being a complete and satisfactory result as it stands. It would be of interest to apply the algebraic methods used in the pebble movers' problem to special cases of the general geometric movers' problem which may admit an algebraic approach.

2. could stand a number of refinements.

a. Find exact constants in the O-terms.

b. It would be useful to at least have an efficient algorithm which approximates the number of moves required. For it seems that only a small fraction of the graph puzzles actually require $O(n^3)$ moves. As an example, it is not hard to show that the "15-puzzle" generalized to square grids of arbitrary size (with one blank) requires only $O(n^{3/2})$ moves (where n is the number of vertices).

3. is only a first step towards understanding the diameter of groups generated by arbitrary cycles. A number of related questions are open:

a. Is the upper bound in 3. tight? Is there a corresponding lower bound of $O(2^{O\sqrt{p}})$ for some instances of 3. ? This would settle the following well-known open problem:

b. Can a transitive group have larger than polynomial diameter for some generator set? Can this be the case for A_n or S_n?

c. Can the upper bound in 3. be generalized to less restrictive conditions on the generating cycles? Is it even true that the following conjecture holds?:

d. Is the diameter of a group, relative to any generating set, always bounded above by $O(n^{\sqrt{n}})$? E.g. the group generated by $S = \{(12)(345)...(...[\text{sum of first } n \text{ primes}])\}$ has diameter $O(2^{\sqrt{n}})$, which satisfies the conjecture.

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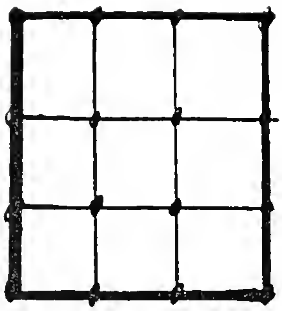

Figure 1
"15-puzzle" graph

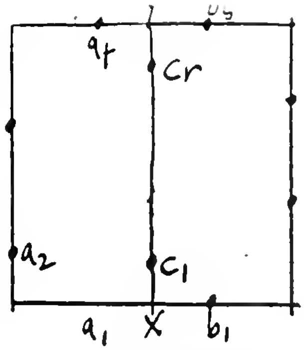

Figure 2
T2-graph

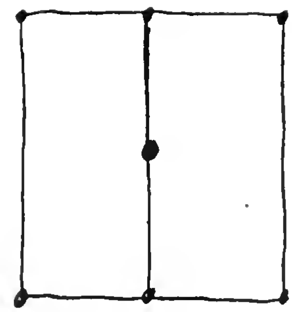

Figure 3
The graph TO

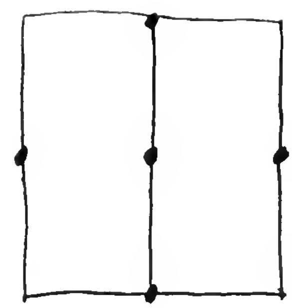

Figure 4
The graph G_1

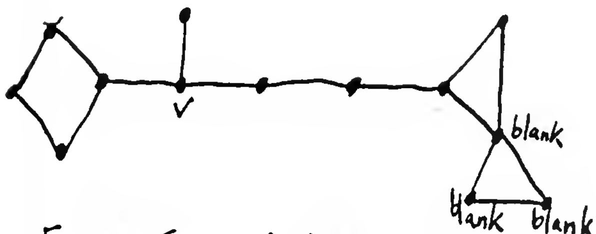

Figure 5 : A tree connecting biconnected components.

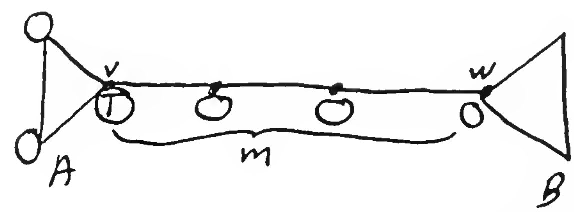

Figure 6 B requires m-1 blanks so that Ⓣ may reach w.

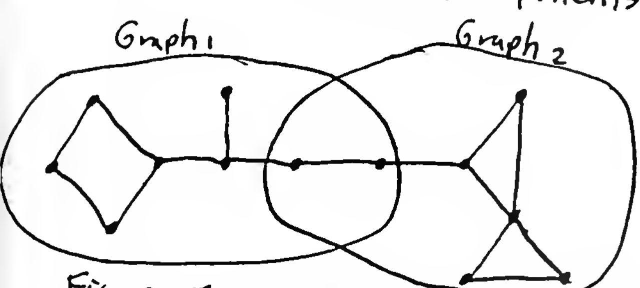

Figure 7 : Decomposition into subgraphs

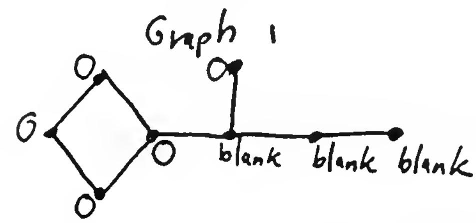

Figure 8 The tokens can move anywhere in Graph 1.

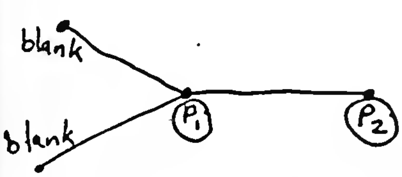

Figure 9 Two blanks enable swap of P_1, P_2

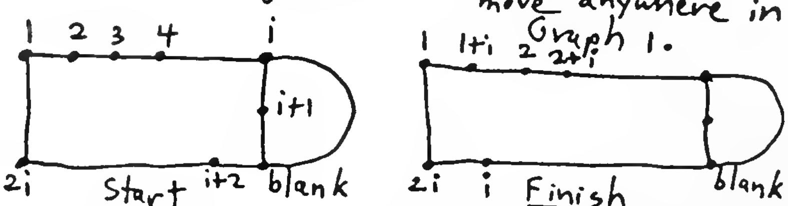

Figure 10 $O(n^3)$ lower bound

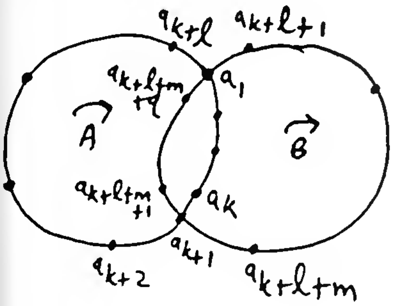

Figure 11
The Hungarian Rings

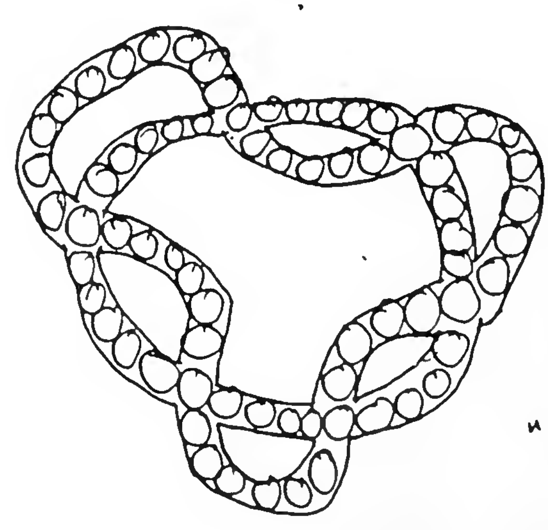

Figure 12
Hungarian ring with multiple crossings - candidate for "exponential" puzzle.